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(INSTITUTE OF MATHEMATICAL SCIENCES)

IAS/IFoS MATHEMATICS by K. Venkanna

IMS

MATHEMATIC OPTIONAL

BY

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Set-I

* Groups *

practice problems

1. Let $(G,*)$ be a group and a be an element of G such that $o(a)=n$. (i) If $a^m=e$ for some positive integer m, then n divides m.

(ii) For every positive integer t,

$$o(a^t) = \frac{n}{\gcd(t,n)}$$

2. which of the following groupoids are semigroups? which are groups?

(i) $(\mathbb{N},*)$ where $a*b=ab$ for all $a,b\in\mathbb{N}$.

(ii) $(\mathbb{N},*)$ where $a*b=b$ for all $a,b\in\mathbb{N}$

(iii) $(\mathbb{Z},*)$ where $a*b=a+b+2$ for all $a,b\in\mathbb{Z}$

(iv) $(\mathbb{Z},*)$ where $a*b=a-b$ for all $a,b\in\mathbb{Z}$.

(v) $(\mathbb{Z},*)$ where $a*b=a+b+ab$ for all $a,b\in\mathbb{Z}$.

(vi) $(\mathbb{R},*)$ where $a*b=a|b|$ for all $a,b\in\mathbb{R}$.

(vii) $(\mathbb{R},*)$ where $a*b=2^ab$ for all $a,b\in\mathbb{R}$

(viii) $(\mathbb{R}\setminus\{-1\},*)$ where $a*b=a+b+ab$ for all $a,b\in\mathbb{R}\setminus\{-1\}$

3. write all complex roots of $x^6=1$. show that they form a group under the usual complex multiplication.

4. Let $G=\{a\in\mathbb{R} : -1<a<1\}$. Define $*$ on G by $a*b=\dfrac{a+b}{1+ab}$ for all $a,b\in G$. show that $*$ is a binary operation on G. Hence Prove that $(G,*)$ is a group.

5. write down the Cayley table for the group operation of the group $\mathbb{Z}_5$.

6. Consider the group $\mathbb{Z}_{30}$. Find the smallest positive integer n such that $n[5]=[0]$ in $\mathbb{Z}_{30}$.

7. write down all elements of the group U_{10} write the Cayley table for this group.

8. Let $G=\left\{\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix} \mid n\in\mathbb{Z}\right\}$. show that G becomes a group under usual matrix multiplication.

9. Find the order of $[6]$ in the group $\mathbb{Z}_{14}$ and the order of $[3]$ in U_{14}.

10. Let $(G,*)$ be a group and $a,b\in G$. Suppose that $a^2=e$ and $a*b*a=b^7$. Prove that $b^{48}=e$.

11. which of the following groupoids are semigroups? which are groups?

(a) $(\mathbb{N},*)$, where $a*b=a+b$ for all $a,b\in\mathbb{N}$

(b) $(\mathbb{N},*)$, where $a*b=a$ for all $a,b\in\mathbb{N}$.

(c) $(\mathbb{Z},*)$, where $a*b=a+b+1$ for all $a,b\in\mathbb{Z}$.

(d) $(\mathbb{Z},*)$, where $a*b=a+b-1$ for all $a,b\in\mathbb{Z}$.

(e) $(\mathbb{Z},*)$, where $a*b=a+2b$ for all $a,b\in\mathbb{Z}$.

(f) $(\mathbb{Z},*)$, where $a*b=a+b-ab$ for all $a,b\in\mathbb{Z}$.

(g) $(\mathbb{R},*)$, where $a*b=|a|b$ for all $a,b\in\mathbb{R}$.

(h) $(\mathbb{R},*)$, where $a*b=a^2b^2$ for all $a,b\in\mathbb{R}$.

(i) $(\mathbb{R},*)$, where $a*b=a+b+ab$ for all $a,b\in\mathbb{R}$.

(j) $(\mathbb{Q}^+,*)$, where $a*b=ab$ for all $a,b\in\mathbb{Q}^+$.

(k) $(\mathbb{Q}\setminus\{0\},*)$, where $a*b=ab$ for all $a,b\in\mathbb{Q}\setminus\{0\}$.

IAS
PREVIOUS YEARS QUESTIONS (2021-1983) SEGMENT-WISE

3 DIMENSIONAL GEOMETRY

2021

- Find the equation of the cylinder whose generators are parallel to the line $x = -\frac{y}{2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1, z = 0$. [10]
- Show that the planes, which cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators, touch the cone $\frac{x^2}{b+c} + \frac{y^2}{c+a} + \frac{z^2}{a+b} = 0$. [20]
- A sphere of constant radius r passes through the origin O and cuts the axes at the points A, B and C. Find, the locus of the foot of the perpendicular drawn from O to the plane ABC. [15]
- Find equation of the plane containing the lines

  $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$,

  $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$.

  Also find the point of intersection of the given lines. [15]

2020

- Find the equation of the tangent plane to the ellipsoid $2x^2 + 6y^2 + 3z^2 = 27$ which passes through the line $x - y - z = 0 = x - y + 2z - 9$. [10]
- Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 4, z = 2$. [15]
- If the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $2yz - 8zx - 3xy = 0$, the find the equations of the other two generators. [15]
- Find the locus of the point intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$. [15]

2019

- Show that the lines

  $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$

  intersect. Find the coordinates of the point of intersection and the equation of the plane containing them. (10)
- (i) The plane $x + 2y + 3z = 12$ cuts the axes of co-ordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC. (10)

  (ii) Prove that the plane $z = 0$ cuts the enveloping cone of the sphere $x^2 + y^2 + z^2 = 11$ which has the vertex at (2, 4, 1) in a rectangular hyperbola. (10)
- Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2 + y^2 = 2az$, but if the point lies on the surface $27a(x^2 + y^2) + 8(a - z)^3 = 0$ then two of the three normals coincide. (15)
- Find the length of the normal chord through a point P of the ellipsoid

  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

  and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the xy-plane, then P lies on the cone.

  $\frac{x^2}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0$ [15]

2018

- Find the projection of the straight line

  $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1}$

  on the plane $x + y + 2z = 6$. (10)
- Find the shortest distance from the point (1,0) to the parabola $y^2 = 4x$. (13)
- The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the x-axis. Find the volume of the solid of revolution. (13)

at its points of intersection with the plane $lx+my+nz = p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2 \quad (15)$$

- Find the equations of the two generating lines through any point $(a\cos\theta, b\sin\theta, 0)$, of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0,$ of the hyperboloid by the plane $z=0$. (15)

2013

- Find the equation of the plane which passes through the points (0,1,1) and (2,0,–1) and is parallel to the line joining the points (–1,1,–2), (3,–2,4). Find also the distance between the line and the plane. (10)
- A sphere S has points (0, 1, 0), (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle. (10)
- Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2(x^2+y^2+z^2)=3r^2$. (15)
- A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0, z=0$ and passes through a fixed point (0, 0, c). If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle
$x^2+y^2+z^2+2ax+2by=0$
$2ax+2by+cz=0$. (15)
- A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2c^2=1$ in P and P'. Prove that $BP.B'P'=a^2+c^2$. (20)

2012

- Prove that two of the straight lines represented by the equation
$$x^3+bx^2y+cxy^2+y^3=0$$
will be at right angles, if $b+c=-2$. (12)
- A variable plane is parallel to the plane
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$
and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone
$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$
(20)
- Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2+y^2+2z=0$ is $x^2+y^2+4z=1$ (20)

2011

- Find the equations of the straight line through the point (3,1,2) to intersect the straight line $x+4=y+1=2(z-2)$ and parallel to the plane $4x+y+5z=0$. (10)
- Show that the equation of the sphere which touches the sphere $4(x^2+y^2+z^2)+10x-25y-2z=0$ at the point (1,2,-2) and passes through the point (-1,0,0) is $x^2+y^2+z^2+2x-6y+1=0$. (10)
- Three points P,Q,R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P,Q,R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere. (20)
- Show that the cone $yz+zx+xy=0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles, and find their area. (20)
- Show that the generators through any one of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60^0 if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other. (20)

2010

- Show that the plane $x+y-2z=3$ cuts the sphere $x^2+y^2+z^2-x+y=2$ in a circle of radius 1 and find the equation of the sphere which has this circle as a great circle. (12)
- Show that the plane $3x+4y+7z+\frac{5}{2}=0$ touches the paraboloid $3x^2+4y^2=10z$ and find the point of contact. (20)

Linear programming *

Set-I

Introduction:

The linear programming originated during world war II (1939–1945), when the British and American Military management called upon a group of scientists to study and plan the war activities, so that maximum damages could be inflicted on the enemy camps at minimum cost and loss. Because of the success in military operations, it quickly spread in all phases of industry and government organisations.

It was first coined in 1940 by Mc Closky and Treften (by using the term operations Research) in a small town, Bowdsey, of the United Kingdom.

In India, it came into existence in 1949, with opening of an operations research unit at the regional research laboratory at Hyderabad.

$$\frac{d}{ad-bc} \cdot \frac{a}{ad-bc} - \frac{-b}{ad-bc} \cdot \frac{-c}{ad-bc} = \frac{1}{ad-bc} \neq 0,$$

we have

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \in G.$$

Now,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence,

G is a group. Now

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in G$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Hence, G is a noncommutative group.

This group is known as the general linear group of degree 2 over $\mathbb{R}$ and is denoted by $GL(2, \mathbb{R})$.